it points in the $\mathbf{i}_z$ direction, so the net viscous torque is

$$\mathcal{T} = \mathbf{i}_z \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} R^2 \sin\theta d\theta d\phi [R\sin\theta][3\eta\Omega\sin\theta]$$
$$= \mathbf{i}_z 8\pi\eta\Omega R^3.$$

Of course, here $\omega = \mathbf{i}_z\Omega$ is the asymptotic angular velocity of the water with respect to the sphere at rest, while $\omega_s = -\mathbf{i}_z\Omega$ is the angular velocity of the sphere with respect to the asymptotically resting water, so the torque on the moving sphere is

$$\mathcal{T}_s = -\omega_s 8\pi\eta R^3. \tag{B26}$$

Thus, in Eq. (A8), $\beta' = 8\pi\eta R^3$.

9. Torque on Ellipsoid

When the shape of the object is the ellipsoid $z^2/a^2 + (x^2+y^2)/b^2 = 1$ $(a \geq b)$, the expression for the torque is of the same form as (B26), but the radius R is replaced by another expression. For rotation about the long (a) axis, the result is[79]

$$R_{\text{eff}}^{-3} = \frac{3}{2}\int_0^\infty d\lambda \frac{1}{(a^2+\lambda)^{1/2}(b^2+\lambda)^2}$$
$$= \frac{3}{2}\Bigg[\frac{a}{(a^2-b^2)b^2}$$
$$- \frac{1}{2(a^2-b^2)^{3/2}} \ln\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\Bigg]. \tag{B27}$$

For $b \to a$, this becomes $a^{-3} + (6/5a^5)(a^2-b^2) + ...$ A good fit for $.2 \leq b/a \leq 1$ is $R_{\text{eff}} \approx .84b + .16a$. With $b = a/2$, (B27) gives $R_{\text{eff}} \approx .59a$.

For rotation about either of the other axes,

$$R_{\text{eff}}^{-3} = \frac{3}{2}\int_0^\infty d\lambda \frac{1}{(a^2+\lambda)^{3/2}(b^2+\lambda)^2}[\lambda + \frac{2a^2b^2}{a^2+b^2}]$$
$$= \frac{3}{2(a^2+b^2)}\Bigg[\frac{-a}{(a^2-b^2)}$$
$$+ \frac{(a^2-(1/2)b^2)}{(a^2-b^2)^{3/2}} \ln\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\Bigg]. \tag{B28}$$

For $b \to a$, this becomes $a^{-3} + (9/10a^5)(a^2-b^2) + ...$ A good fit for $.2 \leq b/a \leq 1$ is $R_{\text{eff}} \approx .56b + .44a$. With $b = a/2$, (B28) gives $R_{\text{eff}} \approx .72a$.

APPENDIX C: GEOMETRICAL OPTICS FROM THE WKB APPROXIMATION

1. The Problem To Be Solved

The problem addressed in the next four appendices is to find the image of a point source of light, made by a ball lens of limited aperture. This is used to discuss the optimal choice of aperture radius.

Consider a ball lens of radius R (diameter D) and index of refraction $n = 3/2$. ($n = 1.5$ is close enough to the BK7 glass index $n = 1.517$ of the ball lens of our experiments.) It follows from Eq. (7) that the focal length of the lens is $f = 3R/2$.

The point source of light has wave-number $k \equiv 2\pi/\lambda$, where $\lambda = .55\mu m$ (green light). It is placed at the focal distance f from the center of the lens. Rays pass through the lens and then through a coaxial hole of radius b in a screen (the so-called exit pupil), and proceed onwards.

The light does not converge to a point at infinity, as predicted by geometrical optics for an ideal lens. Instead, the light intensity distribution which appears on a screen at infinity (placed such as to make the image as sharp as possible) is a circular blob, Although it sounds like something used by a racetrack oddsmaker, this light intensity distribution is called the *point spread function* because it describes how the light from a point source is spread out by the lens.

But, first, the connection should be made between this problem and the one we actually want to solve. The latter is to use the lens as a magnifying glass. That is, one places the point source on the optic axis slightly closer to the lens than f, so that the sharpest image is on a plane at 25cm on the *same* side of the lens as the source. One then divides this magnified image intensity by the lens magnification, $m \approx 25/f$ ($f << 25$cm), to obtain the apparent image intensity to scale (i.e., as if there was in fact a spread-out object of that size being precisely imaged, instead of a point source being imprecisely imaged.)

However, if the source is instead put at the focal length, with the image at $-\infty$, the angular magnification is still m. But, this is the same angular magnification as when the image is at $+\infty$, on the *opposite* side of the lens from the source. It is this simpler problem we are addressing.

The point spread function for this simpler problem can be readily utilized to find the intensity distribution for the magnifier application. For example, suppose we find a ring of light at infinity, with a dark boundary which makes angle β_0 with a point at the center of the lens. The magnifier usage has this circle of vanishing intensity appearing on the 25cm image plane with radius $25\beta_0$. Therefore, the apparent radius of the circle of light is $25\beta_0/m = \beta_0 f$.

2. Light Field in the WKB Approximation

We shall accept the argument[82] that there is no appreciable error in calculating the light intensity by taking the monochromatic light amplitude to be described by a complex scalar field $U(\mathbf{x})\exp -i\omega t$, instead of the actual vector electromagnetic field, with the time average intensity given by $|U(\mathbf{x})|^2$.

The wave equation for $U(\mathbf{x}, t)$ with a point source of light of constant amplitude C at the origin is

$$\nabla^2 U(\mathbf{x},t) - \frac{n^2(\mathbf{x})}{c^2}\frac{\partial^2}{\partial t^2}U(\mathbf{x},t) = -4\pi\delta(\mathbf{x})C.$$

We permit the speed of light $c/n(\mathbf{x})$ to vary throughout space except in the neighborhood of the origin where the speed is that of vacuum, c, and the index of refraction $n(\mathbf{x}) = 1$. Later we shall specialize to $n(\mathbf{x})$ describing a ball lens, i.e., $n(\mathbf{x}) = 1$ everywhere except within a sphere where $n(\mathbf{x}) = n$ is constant. With $U(\mathbf{x},t) = U(\mathbf{x})\exp -i\omega t$ and $\omega = kc$, the wave amplitude is the solution of

$$\nabla^2 U(\mathbf{x}) + k^2 n^2(\mathbf{x})U(\mathbf{x}) = -4\pi\delta(\mathbf{x})C. \qquad \text{(C1)}$$

Away from the source, Eq. (C1) may be written in Schrödinger-like form

$$-\nabla^2 U(\mathbf{x}) - k^2[n^2(\mathbf{x}) - 1]U(\mathbf{x}) = k^2 U(\mathbf{x})$$

It is amusing that turn-of-the-last century physicists got their insights into quantum theory through optics, while we get our insights into optics through quantum theory. The quantum problem, analogous to our optical ball lens problem, is for a particle of mass 1/2 and momentum magnitude k ($\hbar = 1$) emerging from a point and scattering from a spherical potential well of radius R and constant depth $k^2[n^2 - 1]$, The wave function is to have the form $r^{-1}\exp ikr$ in the neighborhood of $r = 0$, a point which lies a distance f from the center of the well. Although this is an exactly solvable problem, it is difficult to obtain physical results from the analytic expression, which is expressed as an infinite series of partial waves[83].

For this reason we shall apply the approximate WKB method. Quantum texts seem to universally discuss this method for one-dimensional motion only. However, the three dimensional problem has been treated[84].

We write $U = \exp i[k\Phi_0 + \Phi_1 + k^{-1}\Phi_2 + ...]$ and substitute into Eq. (C1). Gathering terms of like powers in k, we obtain (away from $\mathbf{x} = 0$ which shall be handled later):

$$\begin{aligned} k^2\nabla\Phi_0\cdot\nabla\Phi_0 &= k^2 n^2, && \text{(C2)} \\ 2k\nabla\Phi_1\cdot\nabla\Phi_0 &= ik\nabla^2\Phi_0 && \text{(C3)} \end{aligned}$$

Eq. (C2) implies that

$$\nabla\Phi_0(\mathbf{x}) = n(\mathbf{x})\hat{\mathbf{v}}(\mathbf{x}), \quad \hat{\mathbf{v}}(\mathbf{x})\cdot\hat{\mathbf{v}}(\mathbf{x}) = 1. \qquad \text{(C4)}$$

To find $\hat{\mathbf{v}}(\mathbf{x})$ requires implementing the restriction that Eq. (C4) is a gradient. To do that, consider

$$\begin{aligned} [\hat{\mathbf{v}}\cdot\nabla]n\hat{\mathbf{v}} &= [\hat{\mathbf{v}}\cdot\nabla]\nabla\Phi_0 = \sum_{i=1}^{3}\hat{v}^i\nabla\frac{\partial}{\partial x^i}\Phi_0 \\ &= \sum_{i=1}^{3} v^i\nabla n v^i = \frac{1}{2n}\nabla[n^2\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}] = \nabla n \qquad \text{(C5)} \end{aligned}$$

Imagine space filled with the vector field $\hat{\mathbf{v}}$. Picture a line passing through a point parallel to the vector $\hat{\mathbf{v}}$ at that point, and continuing on parallel to the vectors it encounters on its path. Such lines are like the flow lines of a fluid in steady state flow, with $\hat{\mathbf{v}}(\mathbf{x})$ as the (constant speed) velocity field.

Now, if one imagines moving along a particular flow line with the fluid, the rate of change of any function $f(\mathbf{x})$ is given by the substantial derivative

$$\frac{D}{Dt}f(\mathbf{x}) \equiv \frac{f(\mathbf{x}+\hat{\mathbf{v}}dt) - f(\mathbf{x})}{dt} = [\hat{\mathbf{v}}\cdot\nabla]f(\mathbf{x}).$$

We see that the left side of Eq. (C5) is the substantial derivative of $n(\mathbf{x})\hat{\mathbf{v}}(\mathbf{x})$.

So, for a fictitious particle of fluid moving with velocity $\hat{\mathbf{v}}(t)$ along a single flow line $\mathbf{x}(t)$, according to Eq. (C5), it satisfies the equation of motion

$$\begin{aligned} \frac{d}{dt}[n(\mathbf{x}(t))\hat{\mathbf{v}}(t)] &= \frac{D}{Dt}n(\mathbf{x})\hat{\mathbf{v}}(\mathbf{x}) = \nabla n(\mathbf{x}(t)) \quad \text{or} \\ \frac{d}{dt}\hat{\mathbf{v}} &= -\hat{\mathbf{v}}\frac{d}{dt}\ln n + \nabla\ln n. \qquad \text{(C6)} \end{aligned}$$

Given a surface, once one specifies the initial velocity vectors on it, the Newton-type law Eq. (C6) then gives the velocity of the fluid elsewhere.

The force in Eq. (C6) ensures that the particle keeps moving with constant speed: the scalar product of Eq. (C6) with $\hat{\mathbf{v}}(t)$ is

$$\frac{1}{2}\frac{d}{dt}[\hat{\mathbf{v}}(t)\cdot\hat{\mathbf{v}}(t)] = [1 - \hat{\mathbf{v}}(t)\cdot\hat{\mathbf{v}}(t)]\frac{d}{dt}\ln n(\mathbf{x}(t))$$

($\hat{\mathbf{v}}\cdot\nabla = \sum_i[dx^i/dt][\partial/\partial dx^i] = d/dt$ has been used), so if $\hat{\mathbf{v}}\cdot\hat{\mathbf{v}} = 1$ initially, that speed is maintained.

It is easily seen that Snell's law is obtained as a consequence of Eq. (C6). If a particle moves in a medium with constant $n = n_1$ (a straight line trajectory since the force vanishes) and passes through a plane interface beyond which $n = n_2$, it receives an impulse perpendicular to the plane. Thus, from the first of (C6), the component of $n\hat{\mathbf{v}}$ parallel to the plane does not change:

$$n_1\hat{\mathbf{v}}_{1,||} = n_2\hat{\mathbf{v}}_{2,||} \quad \text{or} \quad n_1\sin\theta_1 = n_2\sin\theta_2,$$

where θ is the angle made by $\hat{\mathbf{v}}$ and the normal to the plane

We wish to find the solution of Eq. (C4) when $\hat{\mathbf{v}}(\mathbf{x})$ is the velocity field whose flow lines are described by Eq. (C6). We shall now see that Φ_0 is the least action for this motion. For, consider the principle of minimizing the particular action

$$\mathcal{A}(\mathbf{x}) \equiv \int_{\mathbf{x}_0}^{\mathbf{x}} dt\mathcal{L}(\mathbf{x}(t),\dot{\mathbf{x}}(t)) \equiv \int_{\mathbf{x}_0}^{\mathbf{x}} dt n(\mathbf{x}(t))[\dot{\mathbf{x}}(t)\cdot\dot{\mathbf{x}}(t)]^{1/2} \qquad \text{(C7)}$$

We note for later use that t can be replaced by any function of t without altering the action. The principle of least action gives rise to Lagrange's equation,

$$\frac{d}{dt}\frac{n(\mathbf{x})\mathbf{v}}{[\mathbf{v}\cdot\mathbf{v}]^{1/2}} = [\nabla n(\mathbf{x})][\mathbf{v}\cdot\mathbf{v}]^{1/2}, \qquad \text{(C8)}$$

where $\mathbf{v} = \dot{\mathbf{x}}(t)$. By choosing $t = s$, where s is the distance travelled by the particle, so $ds^2 = d\mathbf{x}\cdot d\mathbf{x}$, the velocity $\mathbf{v} = d\mathbf{x}/ds$ has speed $\mathbf{v}\cdot\mathbf{v} = 1$. Then, Eq. (C8) is identical to the equation of motion Eq. (C6).

Imagine the vector field passing through an initial surface, labeled by coordinate $s = 0$ and proceeding onward. Since each member of the family of subsequent $s =$ constant surfaces is perpendicular to $\hat{\mathbf{v}}$, we have $\nabla s(\mathbf{x}) = C(\mathbf{x})\hat{\mathbf{v}}(\mathbf{x})$, where C is some function. However,

$$\hat{\mathbf{v}}\cdot\nabla s = \sum_i \frac{dx^i}{ds}\frac{\partial}{\partial x^i}s = \frac{ds}{ds} = 1,$$

so $C = 1$ and $\nabla s = \hat{\mathbf{v}}$.

Then, Eq. (C7) may be written:

$$\mathcal{A}_0(\mathbf{x}) \equiv \int_0^{s(\mathbf{x})} ds\mathcal{L}(\mathbf{x}(s),\hat{\mathbf{v}}(s)) = \int_0^{s(\mathbf{x})} dsn(\mathbf{x}(s)), \tag{C9}$$

where it is understood that $\mathbf{x}$, $\hat{\mathbf{v}}$ depend not only upon s, but also on two other coordinates, say ξ, η, laid out upon the constant s surfaces. So, from (C9),

$$\nabla\mathcal{A}_0(\mathbf{x}) = n(\mathbf{x}(s))\nabla s = n(\mathbf{x}(s))\hat{\mathbf{v}}(s). \tag{C10}$$

This is the same as (C4), so $\Phi_0 = \mathcal{A}_0$.

Φ_0 is called the optical path length. A light ray follows the flow line of the fictitious particle we have been considering but, of course, it moves along that path with the speed of light c/n. So, when a light ray moves through the distance ds, that takes time dt=dsn/c. Thus, according to Eq. (C9), the optical path length Φ_0 is just the integrated time that light takes to go from one place to another, multiplied by c. (The "principle of least time," the idea that the actual path light takes between two points is the path which takes the least time, is due to Fermat in 1662.) As a consequence, all rays of light which have the same phase at the surface $s = 0$ and travel to the surface s have the same phase there. The surface of constant s is called a "wave front."

To complete the WKB approximation, we need to find Φ_1. Setting $\Phi_1 = i\Phi_1^I$ in Eq. (C3), with use of (C4), we have that

$$2n\hat{\mathbf{v}}\cdot\Phi_1^I = 2n\frac{d}{ds}\Phi_1^I = \nabla\cdot(n\hat{\mathbf{v}}) = \frac{d}{ds}n + n\nabla\cdot\hat{\mathbf{v}}.$$

From the second and fourth terms of this equation,

$$\Phi_1^I(\mathbf{x}) = \ln n^{1/2}(\mathbf{x}) + \frac{1}{2}\int_0^{s(\mathbf{x})} ds\nabla\cdot\hat{\mathbf{v}}(\mathbf{x}(s)). \tag{C11}$$

Thus, from Eqs.(C9),(C11), we obtain the WKB approximate solution of the wave equation:

$$U(\mathbf{x}) = n^{-1/2}(\mathbf{x})e^{-\frac{1}{2}\int_0^{s(\mathbf{x})} ds\nabla\cdot\hat{\mathbf{v}}(\mathbf{x}(s))}e^{ik\int_0^{s(\mathbf{x})} dsn(\mathbf{x}(s))} \tag{C12}$$

Eq. (C12) is what shall be used in what follows. It requires specifying an initial surface for $s = 0$. From this, at any point $\mathbf{x}_0$ on this surface, the initial veocity field $\hat{\mathbf{v}}(\mathbf{x}_0)$ can be determined, since it is perpendicular to the surface and of unit length. Then, one can solve the dynamical equation (C6) to obtain the velocity field elsewhere, and find the specific trajectories $\mathbf{x}(s,\mathbf{x}_0)$. This allows calculation of the integrals in (C12), resulting in the WKB solution $U(\mathbf{x})$. If $n(\mathbf{x}_0) = 1$, this solution has $U(\mathbf{x}_0) = 1$. If a solution with any other value $U_0(\mathbf{x}_0)$ on the $s = 0$ surface is desired, it is $U_0(\mathbf{x}_0)U(\mathbf{x})$.

The last factor in Eq. (C12) is well known in optics, as the eikonal or ray approximation. What has been shown here is that it is justified as the WKB approximate solution of the wave equation.

For our problem, of a point source at $\mathbf{x} = 0$, we choose the $s = 0$ surface to be spherical, of infinitesimal radius, centered at $\mathbf{x} = 0$. Therefore, the initial velocity emanates radially out from $\mathbf{x} = 0$. We assume $n = 1$, for at least a small volume around $\mathbf{x} = 0$. Then, by Eq. (C6), $d\mathbf{v}/dt = 0$ so $\hat{\mathbf{v}}(\mathbf{x}) = \mathbf{r}/r = \hat{\mathbf{r}}$, where $\mathbf{r}$ is the radial vector. Since $\nabla\cdot\hat{\mathbf{r}}f(r) = r^{-2}d^2[r^2f(r)]/dr^2$, with $f = 1$ we get $\nabla\cdot\hat{\mathbf{v}} = 2/r$. The distance travelled from $s = 0$, along the velocity field, is $s = r$. Putting this into Eq. (C12) gives, in this volume,

$$U(\mathbf{x}) = \frac{1}{r}e^{ikr}. \tag{C13}$$

This satisfies the wave equation Eq. (C1), with a unit point source at the origin.

APPENDIX D: REFLECTION FROM LENSES AND MIRRORS

This subsection is a diversion from our main argument, and may be skipped. It is here for logical completeness, and to make some pedagogical points.

In applications to optical systems, light, initially in vacuum, encounters an abrupt change of index of refraction, in the form of lenses or mirrors. The latter may be accommodated by setting $n = -\infty$ in the volume of the mirror. This may be understood from the quantum theory analogy, where $n = -\infty$ turns the potential well into an infinite potential barrier.

How good is the WKB approximation in this case? For completely empty space, Eq. (C13) is the exact solution of Eq. (C1). However, in non-empty space, there is an obvious failure when two rays cross. In that case, from (C4), $\nabla\Phi_0 \sim \hat{\mathbf{v}}$ would then have two possible values, which is impossible.

1. Mirrors

This is what occurs at the surface of a mirror. For example, for a plane mirror at $z = 0$, and an incoming plane wave of wave number k and direction $\hat{\mathbf{v}} = \hat{\mathbf{j}}a + \hat{\mathbf{k}}b$, we know the solution of the wave equation. It is the sum